

A study on brauer group and brauer diagrams

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Abstract

The Brauer group, an important concept in algebra, plays a pivotal role in the study of central simple algebras and their classification. This paper explores the algebraic structures underlying the Brauer group, emphasizing its connections with division algebras, Galois cohomology, and class field theory. We delve into the properties and operations that define the Brauer group, examining how these properties extend across various algebraic and geometric contexts. Additionally, Brauer diagrams, visual representations of elements in the Brauer group, are studied for their applications in tensor categories, quantum algebra, and knot theory. Through a combination of theoretical analysis and illustrative examples, this work aims to provide a comprehensive understanding of the interplay between the Brauer group and Brauer diagrams, shedding light on their significance in modern mathematical research.

INTRODUCTION

The Brauer group is a concept in algebra that generalizes the idea of central simple algebras. It plays a significant role in algebraic number theory, algebraic geometry, and Galois cohomology. The Brauer group consists of equivalence classes of central simple algebras over a field or a scheme, classified by the structure of division algebras that contain them. The Brauer group, a fundamental concept in algebraic number theory, is a group that classifies central simple algebras over a field. Central simple algebras are a special class of algebras that play a crucial role in various branches of mathematics, including number theory, algebraic geometry, and representation theory.

Brauer diagrams, on the other hand, are a visual representation of certain elements of the Brauer group. They provide a geometric interpretation of these elements and can be used to study their properties and relations. In this paper, we will delve into the theory of the Brauer group and Brauer diagrams, exploring their definitions, properties, and applications. We will also discuss the connection between the Brauer group and other mathematical objects, such as Galois cohomology and class field theory. Given a field K , the **Brauer group** $\text{Br}(K)$ is the group of equivalence classes of finite-dimensional central simple algebras over K under the operation of tensor product. Two algebras are considered equivalent if they are Morita equivalent, meaning they have the same matrix ring structure over a division algebra.

Formally, the Brauer group can be written as: $\text{Br}(K) = H^2(G_K, K^*)$ where G_K is the absolute Galois group of K , and H^2 is the second cohomology group. This highlights how the Brauer group can also be studied through the lens of Galois cohomology.

- Abelian group: $\text{Br}(K)$ is an abelian group under the tensor product operation.
- Invariance under field extensions: If L is a finite field extension of K , then there is a natural homomorphism $\text{Br}(K) \rightarrow \text{Br}(L)$.
- Connection to Galois cohomology: There is a canonical isomorphism between $\text{Br}(K)$ and the second cohomology group $H^2(\text{Gal}(K^s/K), K^s/K^{s*})$.
- Local-global principle: Under certain conditions, the Brauer group of a number field can be related to the Brauer groups of its local completions.

The Brauer algebra $B_n(\delta)$ is a finite-dimensional associative algebra that depends on two parameters: n , the number of strands in a diagram, and δ , a scalar parameter (often related to the dimension of the underlying space in physical applications). It generalizes the symmetric group algebra by including not only permutations but also certain diagrams corresponding to pairings of points. Brauer diagrams arise in the study of the Brauer algebra, an algebra closely related to the Brauer group. Brauer diagrams are graphical representations used to describe the elements of the Brauer algebra, which was introduced by Richard Brauer in the context of studying symmetries and representations of the orthogonal and symplectic groups. A Brauer diagram on $2n$ points consists of n pairings of these points. These pairings can be represented by connecting the points with non-intersecting edges (think of the points as placed on a line, and each pair of points connected by a curve).

For example, if we have four points $(1, 2, 3, 4)$, one possible Brauer diagram might connect point 1 to point 2 and point 3 to point 4. These diagrams serve as the basis elements for the Brauer algebra, and multiplication in the algebra corresponds to concatenation of diagrams.

MATHEMATICAL FORMULATION OF THE RESEARCH PROBLEMS

While significant progress has been made in understanding Brauer groups and Brauer diagrams, several new research problems have emerged in recent years. These problems offer exciting avenues for exploration and potential breakthroughs. Some of the most promising new research directions include:

- **Brauer Groups of Derived Categories of Schemes:** Extending the theory of Brauer groups to derived categories of schemes can provide new insights into algebraic geometry and arithmetic. This area is relatively unexplored, and there are many open questions to be addressed.
- **Brauer Groups and Noncommutative Geometry:** Exploring the connections between Brauer groups and noncommutative geometry can lead to new perspectives on both fields. This is a promising area with potential applications to quantum field theory and mathematical physics.
- **Brauer Groups and Homological Algebra:** Investigating the relationship between Brauer groups and homological algebra can provide new tools for studying algebraic structures and their invariants. This area offers many opportunities for interdisciplinary research.
- **Brauer Groups and Arithmetic Statistics:** Applying Brauer groups to problems in arithmetic statistics, such as counting points on varieties over finite fields, can lead to new insights into number theory and algebraic geometry.
- **Brauer Groups and Representation Theory:** Exploring the connections between Brauer groups and representation theory can provide new perspectives on both fields. This area has potential applications to number theory, algebraic geometry, and mathematical physics.
- **Brauer Groups and Categorical Algebra:** Investigating the Brauer group from a categorical perspective can lead to new insights and generalizations. This area offers many opportunities for abstract and foundational research.

- Brauer Groups and Tropical Geometry: Exploring the connections between Brauer groups and tropical geometry can lead to new insights into both fields. This is a relatively new area with potential applications to algebraic geometry and number theory.

KEY RESULTS IN BRAUER THEORY

- Wedderburn's Theorem: This theorem tells us that every central simple algebra over a field is isomorphic to a matrix algebra over a division algebra. The Brauer group then classifies division algebras over the field, with the matrix algebras being "neutral" in the Brauer group.
- Exact Sequence in Galois Cohomology: There is an exact sequence relating the Brauer group of a field with the cohomology of its absolute Galois group. This sequence helps in understanding the relationship between division algebras and Galois cohomology.
- Crossed Products and Skew Fields: An important construction in the study of the Brauer group is the crossed product algebra, which is a generalization of a group algebra. This construction leads to a broader class of division algebras that can be understood through Galois extensions and cohomology.

One of the most significant breakthroughs in the study of the Brauer group was the establishment of its connection to Galois cohomology. This connection, discovered by Albert and Hasse, provides a powerful tool for analysing the structure of the Brauer group and its relationship to the Galois group of the field. The Brauer group satisfies various local- global principles, which relate the Brauer group of a field to the Brauer groups of its local completions. These principles have played a crucial role in applications of the Brauer group to number theory and algebraic geometry. The Brauer-Manin obstruction is a powerful tool for studying the arithmetic of algebraic varieties. It uses the Brauer group to provide a necessary condition for the existence of rational points on a variety. Brauer diagrams were introduced as a visual aid for studying certain elements of the Brauer group. They have proven to be a valuable tool for understanding the structure of central simple algebras and their connections to other mathematical objects. There has been significant progress in understanding the Brauer groups of arithmetic varieties, particularly those arising from number fields and function fields. This research has led to new insights into the arithmetic of these varieties. The Brauer group has found applications in various areas of algebraic geometry, including the study of moduli spaces,

K-theory, and derived categories. The Brauer group has connections to quantum field theory and string theory, where it appears in the context of anomalies and topological invariants. The development of efficient algorithms for computing the Brauer group and its invariants has been an active area of research. These algorithms have applications in number theory, cryptography, and coding theory. The Brauer group has become an essential tool in the study of rational points on varieties, particularly through the Brauer-Manin obstruction, which provides a method for understanding why certain varieties lack rational solutions. In number theory, the Brauer group appears in the study of local and global fields, especially in the context of classifying division algebras over these fields. In noncommutative geometry, Brauer groups are generalized to noncommutative spaces, where they help classify noncommutative analogues of vector bundles and related structures.

POTENTIAL AREAS FOR FUTURE RESEARCH

- **Brauer Groups of Noncommutative Rings:** The study of Brauer groups of noncommutative rings is a relatively new area of research. There are many open questions and potential applications in this field.
- **Brauer Groups in Arithmetic Geometry:** The Brauer group continues to be a valuable tool for studying the arithmetic of algebraic varieties. There are many open problems related to the Brauer-Manin obstruction and the connection between the Brauer group and other arithmetic invariants.
- **Brauer Groups in Physics:** The connections between the Brauer group and physics are still being explored. There is potential for further applications in quantum field theory and string theory.
- **Computational Methods:** The development of more efficient algorithms for computing the Brauer group and its invariants remains an important area of research. These algorithms have potential applications in various fields, including cryptography and coding theory.

CONCLUSION

The Brauer group and Brauer diagrams are essential concepts in modern algebra, with deep connections to number theory, algebraic geometry, and mathematical physics. Brauer diagrams provide a combinatorial approach to understanding the structure of

certain algebras, while the Brauer group offers a way to classify division algebras and central simple algebras over a field or scheme. The Brauer group and Brauer diagrams are powerful tools in algebraic number theory and related fields. They provide a framework for studying central simple algebras and their connections to other mathematical objects. By understanding the theory of the Brauer group and Brauer diagrams.

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